Oxford Cambridge and RSA

Monday 13 May 2019 – Afternoon

AS Level Further Mathematics A

Y531/01 Pure Core

Time allowed: 1 hour 15 minutes

MODEL ANSWERS

You must have:
- Printed Answer Booklet
- Formulae AS Level Further Mathematics A

You may use:
- a scientific or graphical calculator

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INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION

- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

Answer **all** the questions.

1 You are given that $z = 3 - 4\mathrm{i}$.

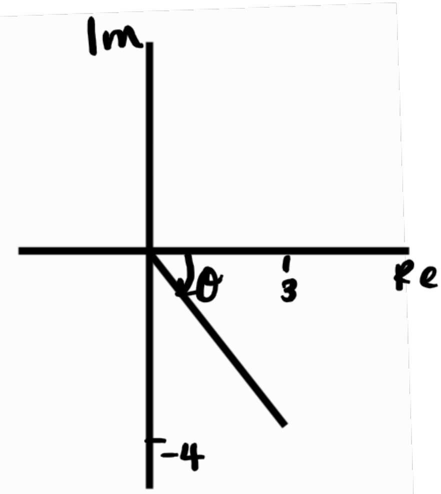
(a) Find
- $|z|$,
- $\arg(z)$,
- z^*. **[3]**

On an Argand diagram the complex number w is represented by the point A and w^* is represented by the point B.

(b) Describe the geometrical relationship between the points A and B. **[2]**

a) • $|z| = \sqrt{3^2+(-4)^2}$

$= 5$



• $\arg z = \tan^{-1}\left(-\frac{4}{3}\right)$

$= -0.927$ rads (3sf)

or

$= -53.1°$

• $z^* = 3+4i$ (conjugate pair)

b) A and B are reflections of eachother in the real axis.

2 Matrices **P** and **Q** are given by $\mathbf{P} = \begin{pmatrix} 1 & k & 0 \\ -2 & 1 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} (1+k) & -1 \end{pmatrix}$ where k is a constant.

Exactly one of statements A and B is true.

Statement A: **P** and **Q** (in that order) are conformable for multiplication.
Statement B: **Q** and **P** (in that order) are conformable for multiplication.

(a) State, with a reason, which **one** of A and B is true. **[2]**

(b) Find either **PQ** or **QP** in terms of k. **[2]**

a) B, the number of columns of the first must equal the number of rows in the second.

$$\begin{pmatrix} (1+k) & -1 \end{pmatrix}\begin{pmatrix} 1 & k & 0 \\ -2 & 1 & 3 \end{pmatrix}$$

$1 \times 2 \qquad 2 \times 3$

(both '2' so they are conformable for multiplication)

b) $QP = \begin{pmatrix} (1+k) & -1 \end{pmatrix}\begin{pmatrix} 1 & k & 0 \\ -2 & 1 & 3 \end{pmatrix}$

$= \begin{pmatrix} (1+k)+2 & k(1+k)-1 & -3 \end{pmatrix}$

$= \begin{pmatrix} (k+3) & (k^2+k-1) & -3 \end{pmatrix}$

3 The position vector of point A is $\mathbf{a} = -9\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$.
The line l passes through A and is perpendicular to $\mathbf{a}$.

(a) Determine the shortest distance between the origin, O, and l. **[2]**

l is also perpendicular to the vector $\mathbf{b}$ where $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

(b) Find a vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. **[1]**

(c) Write down an equation of l in vector form. **[1]**

P is a point on l such that $PA = 2OA$.

(d) Find angle POA giving your answer to 3 significant figures. **[3]**

C is a point whose position vector, $\mathbf{c}$, is given by $\mathbf{c} = p\mathbf{a}$ for some constant p. The line m passes through C and has equation $\mathbf{r} = \mathbf{c} + \mu\mathbf{b}$. The point with position vector $9\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}$ lies on m.

(e) Find the value of p. **[3]**

a) $\sqrt{(-9)^2 + 2^2 + 6^2} = 11$

b) Using the cross product

$$a \times b = \begin{pmatrix} i & j & k \\ -9 & 2 & 6 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ -5 \end{pmatrix}$$

OR Using simultaneous equations

$$\begin{pmatrix} -9 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-9x + 2y + 6z = 0$

let $x = -4$

$-9(-4) + 2y + 6z = 0$

$2y + 6z = -36$ — ①

let $x = -4$

$-2(-4) + y + z = 0$

$y + z = -8$ — ②

$$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-2x + y + z = 0$

$2 \times ② - ①$

$(2y + 2z) - (2y + 6z) = -16 - -36$

$-4z = 20$

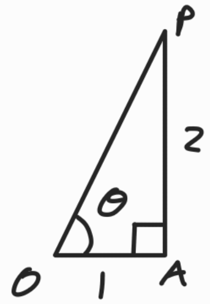
$z = -5$

$\therefore y = -3$

$\therefore$ vector perpendicular $= \begin{pmatrix} -4 \\ -3 \\ -5 \end{pmatrix}$

c) $r = \begin{pmatrix} -9 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$

d) POA is a right angle triangle



$\tan\theta = 2$

$\theta = 1.10714\ldots$

$\theta = 1.11$ rads (3sf)

or

$\theta = 63.4°$ (3sf)

e) $\begin{pmatrix} 9 \\ 8 \\ -12 \end{pmatrix} = pa + \mu b$

$-9p - 2\mu = 9$ — ①

$2p + \mu = 8$ — ②

① + 2 x ②

$-9p - 2\mu + 4p + 2\mu = 9 + 16$

$-5p = 25$

$p = -5$

4 In this question you must show detailed reasoning.

You are given that $f(z) = 4z^4 - 12z^3 + 41z^2 - 128z + 185$ and that $2 + i$ is a root of the equation $f(z) = 0$.

(a) Express $f(z)$ as the product of two quadratic factors with integer coefficients. **[5]**

(b) Solve $f(z) = 0$. **[3]**

Two loci on an Argand diagram are defined by $C_1 = \{z : |z| = r_1\}$ and $C_2 = \{z : |z| = r_2\}$ where $r_1 > r_2$. You are given that two of the points representing the roots of $f(z) = 0$ are on C_1 and two are on C_2. R is the region on the Argand diagram between C_1 and C_2.

(c) Find the exact area of R. **[4]**

(d) ω is the sum of all the roots of $f(z) = 0$.

Determine whether or not the point on the Argand diagram which represents ω lies in R. **[2]**

a) roots: $2+i$, $2-i$

$(z-(2+i))(z-(2-i))$

$= z^2 - 2z + zi - 2z + 4 - 2i - zi + 2i - i^2$

$= z^2 - 4z + 5$

$\Rightarrow (z^2 - 4z + 5)(az^2 + bz + c) \equiv f(z)$

compare coeff:

z^4 coeff: $a = 4$

z^3 coeff: $b - 4a = -12$

$b - 16 = -12$

$b = 4$

constant: $5c = 185$

$c = 37$

$\therefore (z^2 - 4z + 5)(4z^2 + 4z + 37)$

b) $z^2-4z+5=0$: $2+i, 2-i$

$4z^2+4z+37=0$:

$$\frac{-4 \pm \sqrt{4^2-(4\times4\times37}}{2\times4} = \frac{-1\pm 6i}{2}$$

roots: $2+i, 2-i, -\frac{1}{2}+3i, -\frac{1}{2}-3i$

c) $r_2 = |2 \pm i| = \sqrt{5}$

$r_1 = \left|-\frac{1}{2} \pm 3i\right| = \frac{\sqrt{37}}{2}$

$\pi r^2 =$ Area of circle

$$\pi\left(\frac{\sqrt{37}}{2}\right)^2 - \pi\left(\sqrt{5}\right)^2$$

$= \frac{17\pi}{4}$ units^2 (Area between the two circles)

d) $w = -\frac{-12}{4} = 3$

As $\sqrt{5} < 3 < \frac{\sqrt{37}}{2}$, w is in R.

5 In this question you must show detailed reasoning.

You are given that α, β and γ are the roots of the equation $5x^3 - 2x^2 + 3x + 1 = 0$.

(a) Find the value of $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$. **[5]**

(b) Find a cubic equation whose roots are α^2, β^2 and γ^2 giving your answer in the form $ax^3 + bx^2 + cx + d = 0$ where a, b, c and d are integers. **[4]**

a) $\alpha + \beta + \gamma = \frac{-b}{a} = \frac{2}{5}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{3}{5}$

$\alpha\beta\gamma = \frac{-d}{a} = -\frac{1}{5}$

$(\alpha\beta + \beta\gamma + \alpha\gamma)^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2(\alpha^2\beta\gamma + \beta^2\alpha\gamma + \gamma^2\beta\alpha)$

$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \alpha\gamma)^2 - 2(\alpha^2\beta\gamma + \beta^2\alpha\gamma + \gamma^2\beta\alpha)$

$= (\alpha\beta + \beta\gamma + \alpha\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$= \left(\frac{3}{5}\right)^2 - \left(2 \times -\frac{1}{5} \times \frac{2}{5}\right)$

$= \frac{13}{25}$

b) $(\alpha\beta\gamma)^2 = \left(\frac{1}{5}\right)^2 = \frac{1}{25} = -\frac{d}{a}$

$$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2 + 2(\alpha\beta+\alpha\gamma+\beta\gamma)$$

$$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$$

$$= \left(\frac{2}{5}\right)^2 - 2\left(\frac{3}{5}\right)$$

$$= \frac{-26}{25} = \frac{-b}{a}$$

$$\alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\gamma^2 = \frac{13}{25} = \frac{c}{a}$$

$$\therefore \quad x^3 + \frac{26}{25}x^2 + \frac{13}{25}x - \frac{1}{25} = 0 \qquad \times 25$$

$$\underline{25x^3 + 26x^2 + 13x - 1 = 0}$$

6 A transformation T is represented by the matrix $\mathbf{T}$ where $\mathbf{T} = \begin{pmatrix} x^2+1 & -4 \\ 3-2x^2 & x^2+5 \end{pmatrix}$.

A quadrilateral Q, whose area is 12 units, is transformed by T to Q'.

Find the smallest possible value of the area of Q'. **[5]**

Turn over for questions 7 and 8

$$\det T = \begin{vmatrix} x^2+1 & -4 \\ 3-2x^2 & x^2+5 \end{vmatrix}$$

$$= (x^2+1)(x^2+5) - (-4)(3-2x^2)$$

$$= x^4+6x^2+5+12-8x^2$$

$$= x^4-2x^2+17$$

$$= (x^2-1)^2+16$$

when $x=1$, $(x^2-1)^2=0$ (minimum value)

so $\det T = 16$

$\therefore 12 \times 16 = 192$ units

So smallest possible area is 192 units

7 A transformation A is represented by the matrix **A** where $\mathbf{A}=\begin{pmatrix}-1 & x & 2\\ 7-x & -6 & 1\\ 5 & -5x & 2x\end{pmatrix}$.

The tetrahedron H has vertices at O, P, Q and R. The volume of H is 6 units.

P', Q', R' and H' are the images of P, Q, R and H under A.

(a) In the case where $x = 5$

- find the volume of H',
- determine whether A preserves the orientation of H. **[3]**

(b) Find the values of x for which O, P', Q' and R' are coplanar (i.e. the four points lie in the same plane). **[4]**

a) $A=\begin{pmatrix}-1 & 5 & 2\\ 2 & -6 & 1\\ 5 & -25 & 10\end{pmatrix}$

$\det A = -1(-60+25)-5(20-5)+2(-50+30)$
$= 35-75-40 = -80$

So volume of $H' = 6\times 80 = 480$

A does not preserve the orientation $\because$ $\det A < 0$

b) If image is coplanar, $\det A = 0$

$\det A = -1(-12x+5x) - x(2x(7-x)-5)$
$+2(-5x(7-x)+30) = 0$

$\Rightarrow 7x + 2x^3 - 14x^2 + 5x - 70x + 10x^2 + 60 = 0$

$2x^3 - 4x^2 - 58x + 60 = 0$

$(2x-12)(x+5)(x-1) = 0$

$\therefore x = 6 \quad x=-5, \; x=1$

8 **In this question you must show detailed reasoning.**

M is the matrix $\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$.

Prove that $\mathbf{M}^n = \begin{pmatrix} 1 & 3(2^{n+1}-2) \\ 0 & 2^n \end{pmatrix}$, for any positive integer n. **[6]**

<u>when $n=1$</u>

$$M^1 = \begin{pmatrix} 1 & 3(2^{1+1}-2) \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$$

As $M^1 = \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$, it is true when $n=1$.

<u>Assume $n=k$ is true</u>

$$M^k = \begin{pmatrix} 1 & 3(2^{k+1}-2) \\ 0 & 2^k \end{pmatrix}$$

<u>when $n=k+1$</u>

$$M^{k+1} = \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3(2^{k+1}-2) \\ 0 & 2^k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3(2^{k+1}-2)+6\times2^k \\ 0 & 2\times2^k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3(2^{k+1}-2)+3(2^{k+1}) \\ 0 & 2^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3(2^{(k+1)+1} - 2) \\ 0 & 2^{k+1} \end{pmatrix}$$

so true for $n=k$ $\therefore$ must be true for $n=k+1$. But true for $n=1$ $\therefore$ so true for all positive integer n.

END OF QUESTION PAPER